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LAGRANGIAN FINITE ELEMENT METHOD FOR THE ANALYSIS OF TWO-DIMENSIONAL SLOSHING PROBLEMS

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SUMMARY

A new version of a numerical algorithm for the Lagrangian treatment of incompressible fluid flows with free surfaces is developed. The novel features of the present method are the adoptions of the Lagrangian finite element method and the velocity correction technique. The use of the velocity correction approach makes the computational scheme extremely simple in algorithmic structure. Hence, the present method is particularly attractive for large-scale problems. The techniques discussed here are applied to some two-dimensional sloshing problems, which may indicate the versatility and effectiveness of the present method.

INTRODUCTION

This paper deals in detail with a new finite element technique for the numerical solution of fluid flow problems involving free surfaces, and with the numerical analysis of two-dimensional sloshing problems. The analysis of sloshing phenomena is a very important problem of engineering significance, as exemplified by oil oscillations in large storage tanks and water oscillations in reservoirs due to earthquakes. Many theoretical and numerical analyses of sloshing problems have been carried out by many researchers.[1–6] Most of them, however, are based on the potential flow theories, and cannot be applied to oil oscillations with strong viscous effects. To overcome this difficulty the present paper deals with transient motions of incompressible viscous fluid in containers. The numerical analysis of viscous fluid flow problems involving free surfaces is very complicated. There are two reasons for this. First, the position of the free surface varies with time in a manner not known *a priori*, and this fact prevents the analysis from being completely straightforward. Hence, there must be some means of tracking the position of the free surface. Secondly, the accurate free surface boundary condition must be imposed. In solving such difficult problems the finite element method combined with the Lagrangian description can be applied effectively.

Two basic viewpoints are generally considered in discretizing a fluid by a finite difference or finite element method. The first is the Eulerian description, which treats the mesh as a fixed reference frame through which the fluid moves. The second is known as the Lagrangian description, in which the mesh of grid points is embedded in the fluid and moves with it. In the numerical analysis of fluid flows with free surface, a method in which the fluid is described by an Eulerian representation and the free surface is described by a Lagrangian representation is widely employed.[7–11] In such methods it is difficult to solve problems involving complicated free surface structure. Furthermore, it is necessary to discretize the fluid domain at each time step according to the profile of the free

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surface. Hence, it requires a lot of computational time. On the other hand, the Lagrangian updating process is simple in principle, but with large displacements a 'convolution' of the mesh is inevitable. This difficulty can be removed by adopting some automatic rezoning techniques which may help to solve a wide variety of problems. In the following it will be demonstrated how a Lagrangian finite element method may be implemented for an incompressible Newtonian liquid. In the present method the domain is assumed to be covered by a mesh of finite elements whose vertices move with a fluid. In this process, fluid in the interior of a finite element always remains in that element, and fluid boundaries always move with the element boundaries. In an incompressible Lagrangian calculation the volume of each element must remain constant. To satisfy this constraint a velocity correction procedure which is based on Chorin's[12] idea originally used in the finite difference method, is employed. Donea *et al.*[13,14], Schneider and Raithby,[15,16] Glowinsky *et al.*[17], Mizukami and Tsuchiya[18] and Kawahara and Ohmiya[19] have adopted a similar approach to the Eulerian formulation of the Navier–Stokes equations in the case of the finite element method. This method is classified as the semi-implicit method.

BASIC EQUATIONS

Throughout this paper, equations are described by using indicial notation and the summation convention for repeated indices. The problem under consideration is the unsteady motion of a liquid in a two-dimensional rectangular container subjected to forced pitching oscillations. The container has a width $2b$ and is filled with a liquid to a height h in the stationary condition as shown in Figure 1. The rectangular Cartesian co-ordinate system x_i $(i = 1, 2)$ is fixed to the inertial space, and its origin is taken at the centre of the bottom of the container, as shown in Figure 1. Let V be a fluid domain which is surrounded by a piecewise smooth boundary S. The mathematical description of the motion of an incompressible viscous Newtonian fluid is given by the Navier–Stokes equations and the equation of continuity as follows:

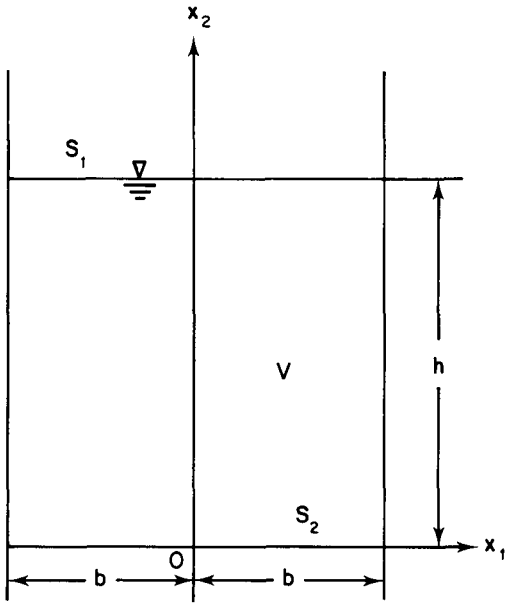


Figure 1. A two-dimensional container

$$\rho \mathrm{D}u_i/\mathrm{D}t - \sigma_{ij,j} + \rho f_i = 0, \qquad \text{in } V \tag{1}$$

$$u_{i,i} = 0, \qquad \text{in } V \tag{2}$$

where

$$\sigma_{ij} = -p\delta_{ij} + 2\mu \mathrm{d}_{ij} \tag{3}$$

$$d_{ij} = (1/2)(u_{i,j} + u_{j,i}) \tag{4}$$

and where $u_i (i = 1, 2)$ are the x_i-components of the fluid velocity; ρf_i are the body forces due to gravity, namely $f_1 = 0$ and $f_2 = g$; p is the pressure; ρ is the constant density; μ is the constant shear viscous coefficient; t is the time. All the parameters used in the calculations are non-dimensionalized by using the width of the container, $2b$, the acceleration due to gravity, g, and the fluid density, ρ, as follows:

$$u_i' = u_i/\sqrt{(2bg)}$$
$$x_i' = x_i/2b$$
$$p' = p/\rho g(2b)$$
$$f_i' = f_i/g$$
$$t' = t\sqrt{(g/2b)}$$

Transforming equations (1) and (2) into non-dimensional form and dropping the primes for simplicity, the equations of motion and the equation of continuity are

$$\mathrm{D}u_i/\mathrm{D}t + p_{,i} - (1/Re)(u_{i,j} + u_{j,i})_{,j} + f_i = 0, \qquad \text{in } V \tag{5}$$

$$u_{i,i} = 0, \qquad \text{in } V \tag{6}$$

where Re is the Reynolds number and is defined as $Re = \rho(2b)\sqrt{(2bg)}/\mu$.

The boundary S consists of two kinds of boundaries, namely the free surface boundary S_1 and the solid wall boundary S_2 of the container. On the free surface boundary, the normal stress should be equal to the atmospheric pressure and the tangential stress should vanish. These conditions are expressed as

$$\sigma_{ij} n_j = 0, \qquad \text{on } S_1 \tag{7}$$

where the atmospheric pressure is assumed to be zero and n_j is the direction cosine of the outward normal on the boundary with respect to x_j-axis. On the wall of the container, the normal component of the fluid velocity to the wall is equal to the transverse velocity of the container due to a pitching motion. Then, the boundary condition is expressed as

$$u_i n_i = r\Omega(t), \qquad \text{on } S_2 \tag{8}$$

where $\Omega(t)$ is the angular velocity of the pitching motion, whose direction is assumed to be positive in the clockwise direction and r is the distance from the origin to the nodal points on the wall of the container. When the amplitude of the pitching motion is small, the boundary condition (8) can be approximated as

$$u_1 = r\Omega(t) \tag{9}$$

$$u_2 = 0 \tag{10}$$

Initial conditions for the present analysis are clearly stated in the section of numerical examples.

LAGRANGIAN METHOD

In order to implement a numerical solution procedure for the Lagrangian formulation it is necessary to discretize time and material. This is in contrast to a numerical solution of an Eulerian formulation, where one would discretize time and space. The basic concept of the analysis presented in this paper is the idea that the material time derivative can be approximately determined by the functions known at both deformed and undeformed positions during a short time increment. The function at the deformed position cannot be obtained before computation. Thus, iterative computation is necessary.

The total time interval is divided into many short time increments, denoted by Δt, and the fluid is discretized into a lot of triangular finite elements. The location of a nodal point of an element is denoted by x_i^n at the nth time instant $t^n (t^n = n\Delta t)$. The velocities at t^n and t^{n+1} are written as

$$u_i^n = u_i(t^n, x_i^n) \tag{11}$$

$$u_i^{n+1} = u_i(t^{n+1}, x_i^{n+1}) \tag{12}$$

The material derivative is the time derivative considering the change of location of a fluid particle. It can be approximated by using equations (11) and (12) as in the following form:

$$Du_i/Dt \doteqdot (u_i^{n+1} - u_i^n)/\Delta t \tag{13}$$

The location of a nodal point after the increment Δt is given by

$$x_i^{n+1} = x_i^n + (\Delta t/2)(u_i^{n+1} + u_i^n) \tag{14}$$

Using equation (14), the location can be determined once the velocity has been computed.

As previously mentioned, it is very important that the boundary conditions on the free surface should be applied accurately. Then, in the present Lagrangian method, an iterative computational procedure is used to obtain the accurate position of the free surface. The iteration is repeated in every time step and proceeds as follows: at the initial step of iteration, velocity $u_i^{n+1(0)}$, pressure $p^{n+1(0)}$ and position $x_i^{n+1(0)}$ are computed as

$$u_i^{n+1(0)} = g_i(u_i^n, p^n, x_i^n) \tag{15}$$

$$p^{n+1(0)} = h(u_i^n, p^n, x_i^n) \tag{16}$$

$$x_i^{n+1(0)} = x_i^n + \Delta t u_i^n \tag{17}$$

where $u_i^{n+1(0)}$, for example, means the value of the velocity component at the initial step of iteration in the $(n+1)$th time interval; g_i and h mean that velocity and pressure can be computed following the velocity correction method (explained in the next section) based on the previously known velocity u_i^n and pressure p^n at position x_i^n and at time t^n. At the mth iteration cycle (where $m = 1, 2, \ldots,$ MAX; MAX is the maximum number of iterations) the values are updated by the following equations:

$$u_i^{n+1(m)} = g_i(u_i^n, p^n, x_i^{n+1(m-1)}) \tag{18}$$

$$p^{n+1(m)} = h(u_i^n, p^n, x_i^{n+1(m-1)}) \tag{19}$$

$$x_i^{n+1(m)} = x_i^n + (\Delta t/2)(u_i^{n+1(m-1)} + u_i^n) \tag{20}$$

Equations (18) and (19) specify that the velocity and pressure can be computed using the latest positions of the fluid particles. The iteration is repeated until the computed velocity satisfies the following convergence criterion:

$$|u_i^{n+1(m)} - u_i^{n+1(m-1)}| < \varepsilon \tag{21}$$

where ε is a previously defined small value. At convergence, it is obtained that

$$u_i^{n+1} = u_i^{n+1(m)} \tag{22}$$

$$p^{n+1} = p^{n+1(m)} \tag{23}$$

$$x_i^{n+1} = x_i^{n+1(m)} \tag{24}$$

Thus the velocity and pressure considering the change of position of a fluid particle can be computed. The computation is repeated for all $n = 1, 2, \ldots, \text{NMAX}$, where NMAX is the total number of time points.

The procedure outlined above employs the lagrangian description of fluid motion. Its advantages are that material interfaces are properly maintained and the absence of numerical instability associated with convective terms.

VELOCITY CORRECTION METHOD

To obtain the velocity and pressure at time t^{n+1}, the velocity correction method is successfully used in the present analysis. This section describes how to compute the functions g_i and h in equations (15) and (16), which in turn help to compute u_i and p in equations (18) and (19).

Using equation (13), equation of motion (5) can be discretized into the following form:

$$\tilde{u}_i^{n+1} = u_i^n - \Delta t\{p_{,i}^n - (1/Re)(u_{i,j}^n + u_{j,i}^n)_{,j} + f_i^n\} \tag{25}$$

where $\tilde{u}_i^{n+1}$ is the approximate velocity field not satisfying the incompressibility condition. We denote the exact velocity which satisfies the incompressibility constraint by u_i^{n+1} and pressure by p^{n+1} at time t^{n+1}, and assume that they satisfy the following equations:

$$u_i^{n+1} = u_i^n - \Delta t\{p_{,i}^{n+1} - (1/Re)(u_{i,j}^n + u_{j,i}^n)_{,j} + f_i^n\} \tag{26}$$

$$u_{i,i}^{n+1} = 0 \tag{27}$$

Taking rotation on both sides of equations (25) and (26) the following relation can be obtained:

$$\text{Curl}\, u_i^{n+1} = \text{Curl}\, \tilde{u}_i^{n+1} \tag{28}$$

From the above relation it is clear that

$$u_i^{n+1} = \tilde{u}_i^{n+1} + \phi_{,i} \tag{29}$$

where ϕ is some scalar potential. Taking the divergence on both sides of equation (29) together with the incompressibility constraint (27), the following equation for ϕ can be derived:

$$\phi_{,ii} = -\tilde{u}_{i,i}^{n+1} \tag{30}$$

To solve the above equation the following boundary conditions are applied:

$$\phi = 0, \qquad \text{on } S_1 \tag{31}$$

$$\phi_{,i} n_i = 0, \qquad \text{on } S_2 \tag{32}$$

Equation (30) can be solved by the finite element method. Thus the potential at all nodal points in the flow field can be obtained. The substitution of the resulting potential in equation (29) yields the corrected velocities at all nodal points. Pressure, p^{n+1}, can be calculated as

$$p^{n+1} = p^n - \phi/\Delta t \tag{33}$$

Equation (33) can be derived from equations (25), (26) and (29). Based on the procedures expressed by equations (25), (30), (29) and (33), the velocity u_i^{n+1} and pressure p^{n+1} can be obtained.

FINITE ELEMENT METHOD

Equation (25) with (8); equations (29), (30) with (31), (32); and equation (33) can be discretized by the finite element method. Then recalling equations (11) and (12), one can evaluate all the functions at time t^{n+1} or t^n. The superscript n denotes the function evaluated at time t^n and position x_i^n. A weak formulation of the problem defined by equations (25), (29) and (30) is obtained by multiplying the differential equations by suitable weighting functions and integrating over the domain V. Multiplying the equations (25) and (29) by the weighting function v_i, and equation (30) by the weighting function q and integrating them over the domain V, the following weighted residual equations of the original problem are obtained:

$$\int_{V^{n+1}} (v_i^{n+1}\tilde{u}_i^{n+1})\mathrm{d}V = \int_{V^n} (v_i^n u_i^n)\mathrm{d}V + \Delta t \int_{V^n} (v_{i,i}^n p^n)\mathrm{d}V - (\Delta t/Re)\left\{\int_{V^n} (v_{i,j}^n u_{i,j}^n)\mathrm{d}V + \int_{V^n} (v_{i,j}^n u_{j,i}^n)\mathrm{d}V\right\} - \Delta t \int_{V^n} (v_i^n f_i^n)\mathrm{d}V + \Delta t \int_{S^n} (v_i^n \sigma_{ij} n_j)\mathrm{d}S \tag{34}$$

$$\int_{V^{n+1}} (v_i^{n+1} u_i^{n+1})\mathrm{d}V = \int_{V^{n+1}} (v_i^{n+1}\tilde{u}_i^{n+1})\mathrm{d}V + \int_{V^{n+1}} (v_i^{n+1}\phi_{,i})\mathrm{d}V \tag{35}$$

$$\int_{V^{n+1}} (q_{,i}^{n+1}\phi_{,i})\mathrm{d}V = \int_{V^{n+1}} (q^{n+1}\tilde{u}_{i,i}^{n+1})\mathrm{d}V + \int_{S^{n+1}} (q^{n+1}\phi_{,i} n_i)\mathrm{d}S \tag{36}$$

Assume that the flow field to be analysed is divided into a number of small domains called finite elements. The velocity and pressure in each finite element are interpolated by using linear shape functions and are expressed in the following form:

$$u_i^n = \Phi_\alpha^n u_{\alpha i}^n \tag{37}$$

$$p^n = \Phi_\alpha^n p_\alpha^n \tag{38}$$

where Φ_α^n is the shape function of the αth node at time t^n and location $x_i^n \cdot u_{\alpha i}^n$ denotes the nodal velocity at the αth node in the ith direction at time t^n and location x_i^n and p_α^n is the pressure at the αth node at time t^n. The weighting functions are interpolated in the manner similar to equations (37) and (38) as

$$v_i^n = \Phi_\alpha^n v_{\alpha i}^n \tag{39}$$

$$q^n = \Phi_\alpha^n q_\alpha^n \tag{40}$$

where $v_{\alpha i}^n$ and q_α^n denote the values at the αth node at time t^n. Introducing equations (37)–(40) into equations (34)–(36) and using the arbitrariness of the weighting functions, the finite element equations are derived as follows:

$$\bar{M}_{\alpha\beta}^{n+1}\tilde{u}_{\beta i}^{n+1} = \bar{M}_{\alpha\beta}^n u_{\beta i}^n - \Delta t(S_{\alpha i\beta j}^n u_{\beta j}^n - H_{\alpha i\beta}^n p_\beta^n + N_\alpha^n f_{\alpha i}^n - \hat{\Omega}_{\alpha i}^n) \tag{41}$$

$$\bar{M}_{\alpha\beta}^{n+1} u_{\beta i}^{n+1} = \bar{M}_{\alpha\beta}^{n+1}\tilde{u}_{\beta i}^{n+1} + H_{\alpha i\beta}^{n+1}\phi_\beta \tag{42}$$

$$A_{\alpha\beta}^{n+1}\,\phi_\beta = H_{\alpha i\beta}^{n+1}\,\tilde{u}_{\beta i}^{n+1} + \hat{\Sigma}_\alpha^{n+1} \tag{43}$$

where

$$M_{\alpha\beta}^{n+1} = \int_{V^{n+1}} (\Phi_\alpha^{n+1}\,\Phi_\beta^{n+1})\,\mathrm{d}V$$

$$M_{\alpha\beta}^{n} = \int_{V^{n}} (\Phi_\alpha^{n}\,\Phi_\beta^{n})\,\mathrm{d}V$$

$$H_{\alpha i\beta}^{n+1} = \int_{V^{n+1}} (\Phi_{\alpha,i}^{n+1}\,\Phi_\beta^{n+1})\,\mathrm{d}V$$

$$H_{\alpha i\beta}^{n} = \int_{V^{n}} (\Phi_{\alpha,i}^{n}\,\Phi_\beta^{n})\,\mathrm{d}V$$

$$A_{\alpha\beta}^{n+1} = \int_{V^{n+1}} (\Phi_{\alpha,i}^{n+1}\,\Phi_{\beta,i}^{n+1})\,\mathrm{d}V$$

$$N_\alpha^n = \int_{V^n} (\Phi_\alpha^n)\,\mathrm{d}V$$

$$\hat{\Omega}_{\alpha i}^n = \int_{S^n} (\Phi_\alpha^n \sigma_{ij} n_j)\,\mathrm{d}S$$

$$\hat{\Sigma}_\alpha^{n+1} = \int_{S^{n+1}} (\Phi_\alpha^{n+1}\,\phi_{,i} n_i)\,\mathrm{d}S$$

$$S_{\alpha i\beta j}^n = (1/Re)\left\{\int_{V^n} (\Phi_{\alpha,k}^n\,\Phi_{\beta,k}^n)\delta_{ij}\,\mathrm{d}V + \int_{V^n} (\Phi_{\alpha,i}^n\,\Phi_{\beta,j}^n)\,\mathrm{d}V\right\}$$

In equations (41) and (42), $\bar{M}_{\alpha\beta}^n$ means the lumped matrix obtained from the consistent matrix $M_{\alpha\beta}^n$.

NUMERICAL EXAMPLES

Consider the container as shown in Figure 1, which is subjected to the following forced sinusoidal pitching oscillation about the origin o:

$$\theta(t) = \Theta \cos \omega t; \qquad t \geqslant 0 \tag{44}$$

where $\theta(t)$ is the angle of inclination of the container at time t, and Θ and ω are the amplitude and frequency of the forced oscillation, respectively. The angular velocity of pitching is then given as follows:

$$\begin{aligned}\Omega(t) &= \mathrm{d}\theta(t)/\mathrm{d}t \\ &= -\Theta\omega \sin \omega t; \qquad t \geqslant 0\end{aligned} \tag{45}$$

At $t = 0$, the container is assumed to be inclined at an angle θ and the fluid in the container is at rest. From these assumptions the initial conditions for the present analysis are given as follows:

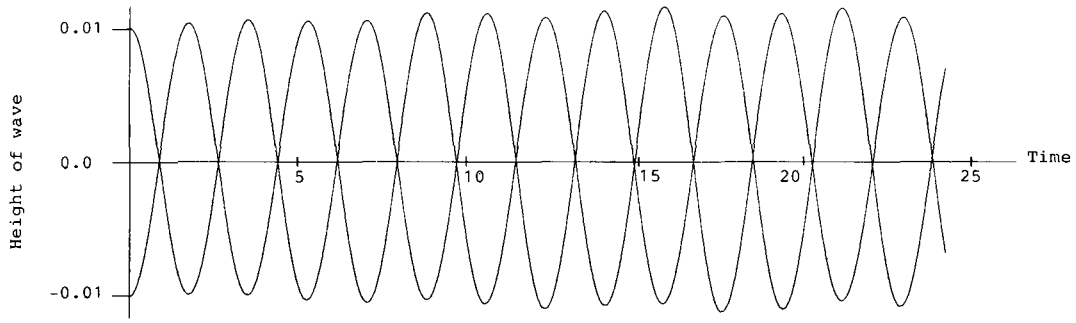


Figure 2. Free oscillation without viscous effect

$$u_i(x_1, x_2, 0) = 0 \tag{46}$$

$$\eta(x_1, 0) = x_1 \tan \Theta + h \tag{47}$$

where $\eta(x_1, t)$ is the displacement of the free surface from the undisturbed free surface. In all the subsequent calculations the total number of nodal points and finite elements are 441 and 800, respectively.

Free oscillation

First, we compute the free oscillation with small amplitude of the liquid in the container to verify the adaptability of the present finite element computer program with Lagrangian description. As the initial profile of the free surface, the first antisymmetric natural mode with amplitude 0·01 is assumed. The time increment is $\Delta t = 0{\cdot}005$. Figure 2 shows the computed time histories of $\eta(\pm 1/2, t)$ in the case of a non-viscous effect. From the computed result it can be seen that the free surface oscillates at a nearly constant amplitude without any artificial damping effect. Figure 3 shows the time histories of $\eta(\pm 1/2, t)$ with a viscous effect, where ν is 0·01.

Forced oscillation without viscous effect

Consider the liquid in a container which is forced to pitch about the origin, as given by equation (44). The parameters used in the calculation are $\theta = 0{\cdot}2°$, $\omega = 1{\cdot}05$, $\nu = 0{\cdot}0$ and $\Delta t = 0{\cdot}002$, where ν is the kinematic viscosity. The computed wave profile, velocity and pressure at elapsed times $t = 12$ and $t = 21$ are shown in Figures 4 and 5, respectively. Figure 6 shows the time histories of $\eta(\pm 1/2, t)$ for the liquid in the container. The computed results show the distribution of swelling, which clearly coincides with the physical phenomena in the case of non-viscous fluid flow analysis.

Forced oscillation with viscous effect

The last example is the computation of a sloshing problem with viscous effect. The parameters used in the calculation are $\theta = 0{\cdot}2°$, $\omega = 1{\cdot}57$, $\nu = 0{\cdot}01$ and $\Delta t = 0{\cdot}002$. Figure 7 shows the time histories of $\eta(\pm 1/2, t)$ for the liquid in the container. The use of viscous fluid gives rise to a damping effect, which makes the displacement of the free surface constant after a certain period of time.

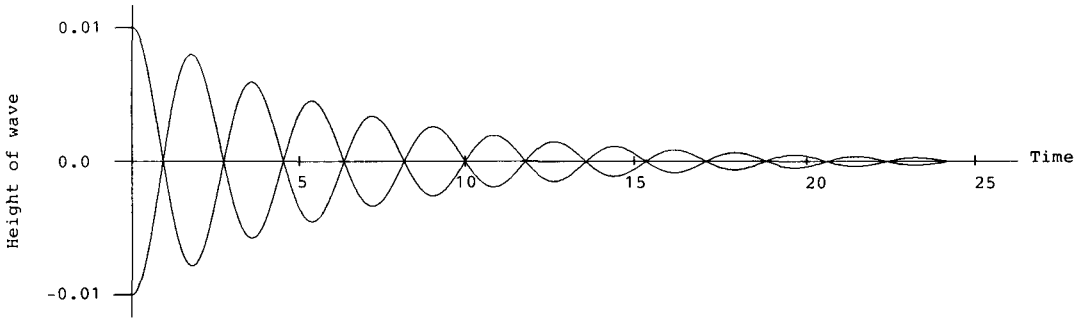
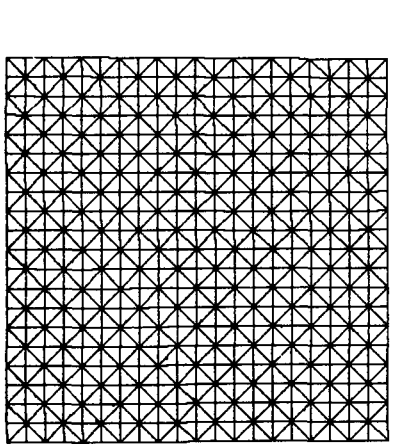
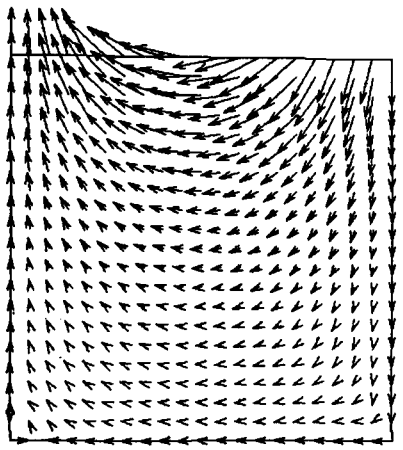


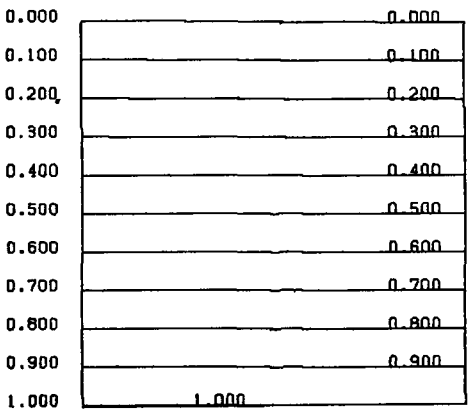
Figure 3. Free oscillation with viscous effect



(a)



(b)



(c)

Figure 4. Computed results at time $t = 12{\cdot}0$: (a) mesh division; (b) velocity distribution; (c) pressure distribution

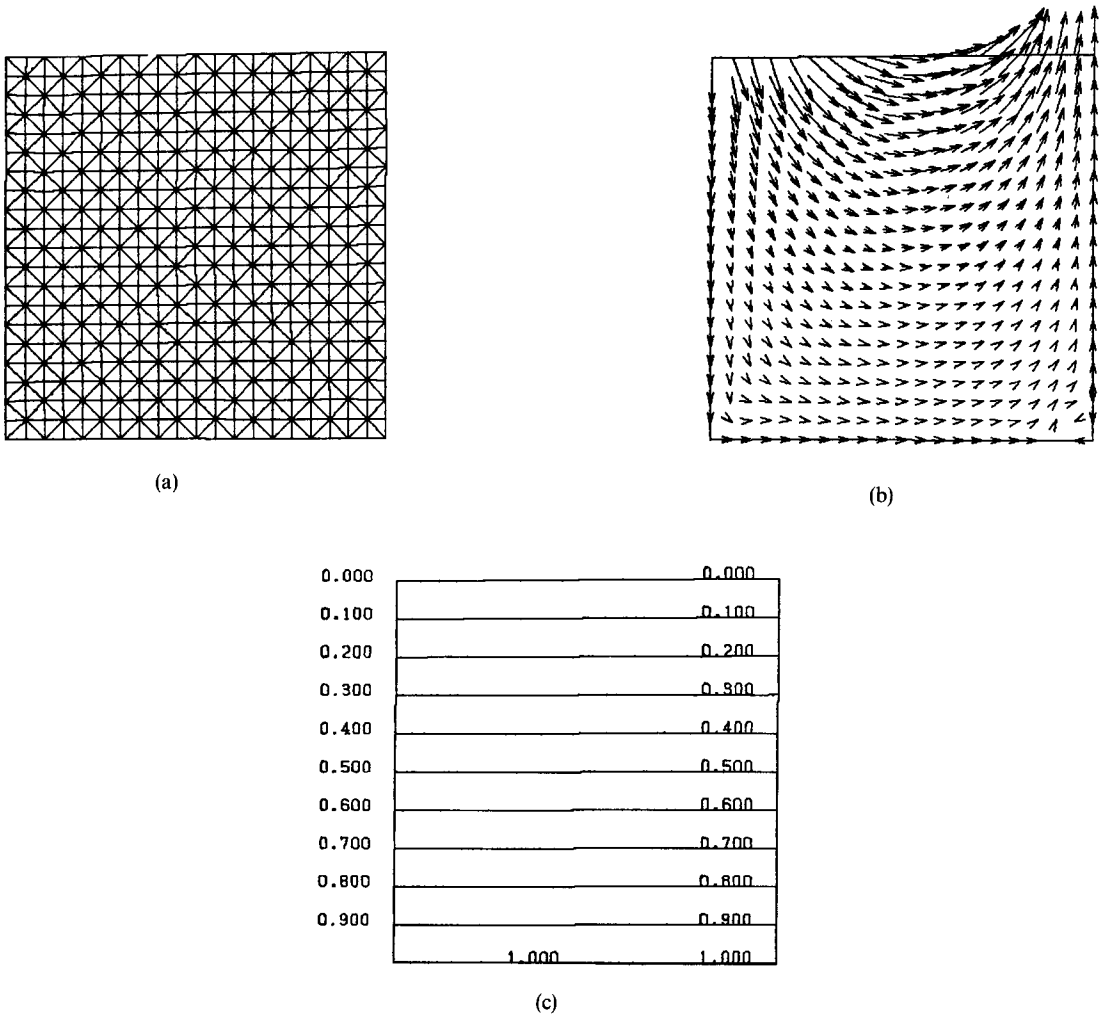


Figure 5. Computed results at time $t = 21{\cdot}0$: (a) mesh division; (b) velocity distribution; (c) pressure distribution

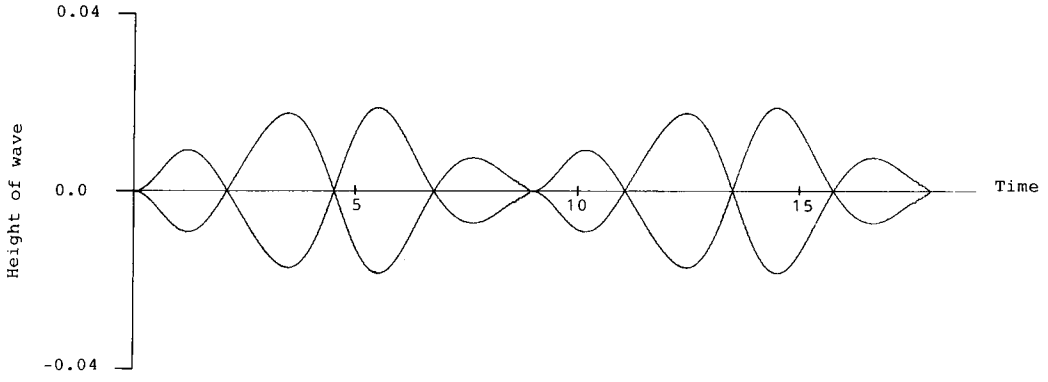


Figure 6. Time histories of $\eta(\pm 1/2, t)$

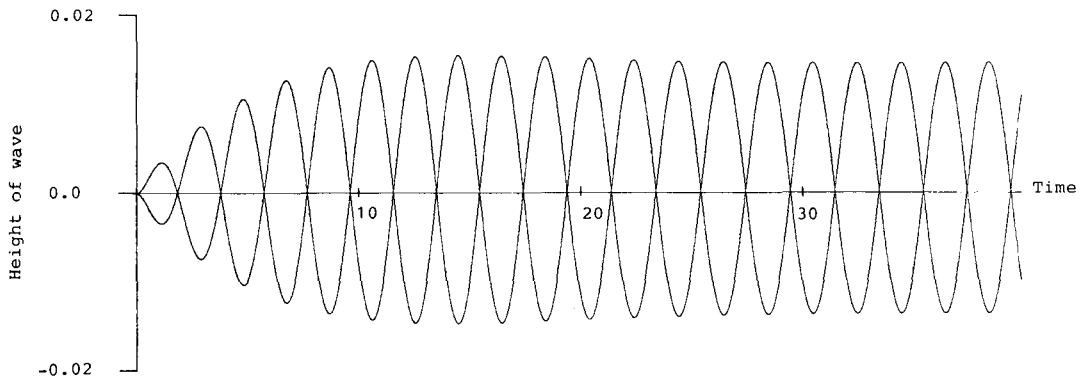


Figure 7. Time histories of $\eta(\pm 1/2, t)$

CONCLUDING REMARKS

The Lagrangian finite element method has been presented for calculating the transient dynamics of incompressible, viscous fluids having free surfaces. This new method possesses an extremely simple algorithmic structure, which is achieved by adopting the velocity correction method for the time integration of the unsteady Navier–Stokes equations. Lagrangian co-ordinates permit a good treatment of free surfaces. The present method is illustrated with some physical problems involving sloshing dynamics of inviscid and viscous fluids, which indicate the possibility of employing the same technique to attack a wide variety of water wave problems. The authors are at present looking at the adaptation of this method to more difficult problems, such as those involving highly non-linear and large viscosity effects.

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